*A Vector Determination of Interplanar Spacings of Crystal Systems**

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THE CALCULATION of the spacings between successive planes in any given type of crystal structure is at present based on a theorem in solid analytic geometry which gives the perpendicular distance d from any point to a plane. For crystal lattices other than the rectangular types, analytical difficulties enter into the derivation of the expression for d, since in these situations the coordinate frame of reference must be oblique rather than rectangular. In addition to the complexity of the derivation, the expression for d is usually given in a complicated and inelegant form.

In the following we offer a derivation of a general formula for d by vector methods. It is believed that this derivation is new. At any rate it is considerably briefer and simpler than that given by analytical methods, and the resulting formula leaves little to be desired by way of compactness and elegance.

FUNDAMENTAL THEOREM

IN A TETRAHEDRON V–ABC, let d be the altitude from V. Then

$$d^2 = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} \div \begin{vmatrix} a' & b' & c' & 0 \\ 1 & \cos\gamma & \cos\beta & a' \\ \cos\gamma & 1 & \cos\alpha & b' \\ \cos\beta & \cos\alpha & 1 & c' \end{vmatrix},$$

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where

$a'=1/VA, b'=1/VB, c'=1/VC$ and $\alpha=\measuredangle BVC, \beta=\measuredangle CVA, \gamma=\measuredangle AVB.$

Let $\bar{a}, \bar{b}, \bar{c}$ be unit vectors along VA, VB, VC respectively, and n the unit vector along d. Let n_a, n_b, n_c be the magnitudes of the components of n along VA, VB, VC respectively. Also, set $a=VA$, $b=VB$, $c=VC$. Then we have

(1) $$\bar{n}=n_a\bar{a}+n_b\bar{b}+n_c\bar{c},$$

(2) $$(\bar{n}.\bar{a})a=(\bar{n}.\bar{b})b=(\bar{n}.\bar{c})c=d,$$

(3) $$\bar{b}.\bar{c}=\cos\alpha,\quad \bar{c}.\bar{a}=\cos\beta,\quad \bar{a}.\bar{b}=\cos\gamma.$$

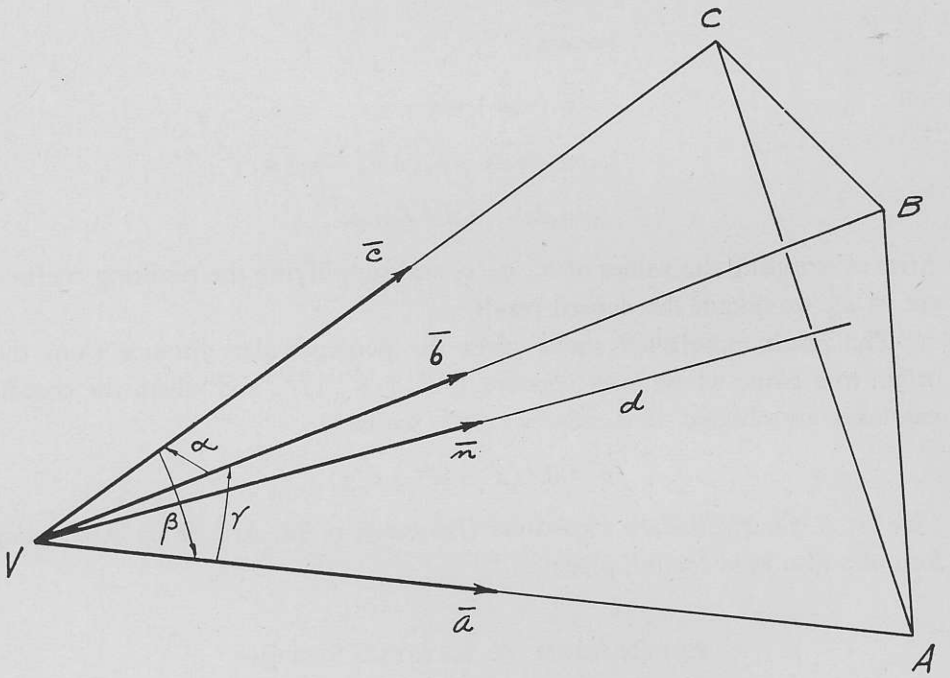

Figure 1

From (1) and (2) we have

$$[(n_a\bar{a}+n_b\bar{b}+n_c\bar{c}).\bar{a}]a=d,$$

or, using (3),

$$n_a+n_b\cos\gamma+n_c\cos\beta=da'.$$

Similarly we have

$$n_a \cos\gamma + n_b + n_c \cos\alpha = db',$$
$$n_a \cos\beta + n_b \cos\alpha + n_c = dc'.$$

Solving for n_a, n_b, n_c by Cramer's rule, we get

$$n_a = d \begin{vmatrix} a' & \cos\gamma & \cos\beta \\ b' & 1 & \cos\alpha \\ c' & \cos\alpha & 1 \end{vmatrix} \div \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}, \text{ etc.}$$

But

$$\begin{aligned} 1 &= \bar{n}.\bar{n} \\ &= \bar{n}.(n_a\bar{a} + n_b\bar{b} + n_c\bar{c} \\ &= n_a(\bar{n}.\bar{a}) + n_b(\bar{n}.\bar{b}) + n_c(\bar{n}.\bar{c}) \\ &= n_a da' + n_b db' + n_c dc'. \end{aligned}$$

After substituting the values of n_a, n_b, n_c and simplifying the resulting coefficient of d^2, we obtain the desired result.

The result established above gives the perpendicular distance from the origin to a plane whose intercepts are $1/a'$, $1/b'$, $1/c'$, and where the coordinate axes are oblique. If $\alpha=\beta=\gamma=90°$, we have

$$d^2 = 1/(a'^2 + b'^2 + c'^2).$$

(See N. A. Court, *Modern Pure Solid Geometry,* p. 92, Art. 285.) Analogous formulas also hold for the plane.

Application to Crystal Systems

LET $d_{hk\,l}$ be the perpendicular distance between the successive planes of a given set of parallel crystal inter-planes where (h, k, l) are the Miller indices of the set of planes. Let a_0, b_0, c_0 be the lattice constants and α, β, γ the lattice angles. Then if, in the above theorem, we replace a', b', c' by h/a_0, k/b_0, l/c_0, the formula will give us $d_{hk\,l}$. Special and simplified formulas for the various crystal types may then be found by utilizing the characteristics as listed in the following table:

LATTICE	CHARACTERISTICS
I. Rectangular	$\alpha = \beta = \gamma \quad = 90^\circ$
a. Cubic	$a_0 = b_0 = c_0$
b. Tetragonal	$a_0 = b_0 \neq c_0$
c. Orthorhombic	$a_0 \neq b_0 \neq c_0$
II. Hexagonal	$\alpha = \beta = 90^\circ, \gamma = 120^\circ$
	$a_0 = b_0 \neq c_0$
III. Monoclinic	$\alpha = \gamma = 90^\circ, \beta \neq 90^\circ$
	$a_0 \neq b_0 \neq c_0$
IV. Rhombohedral	$\alpha = \beta = \gamma \quad \neq 90^\circ$
	$a_0 = b_0 = c_0$
V. Triclinic	$\alpha \neq \beta \neq \gamma \quad \neq 90^\circ$
	$a_0 \neq b_0 \neq c_0$

The distance d between two successive cleavage planes having Miller indices (h, k, l) is, then, given by

$$d = [h, k, l] d_{hkl},$$

where $[h, k, l]$ represents the least common multiple of the integers h, k, l.

Figure 2 on page 22 illustrates the relation between d and d_{hkl} for $(h, k, l) = (6, 3, 4)$.

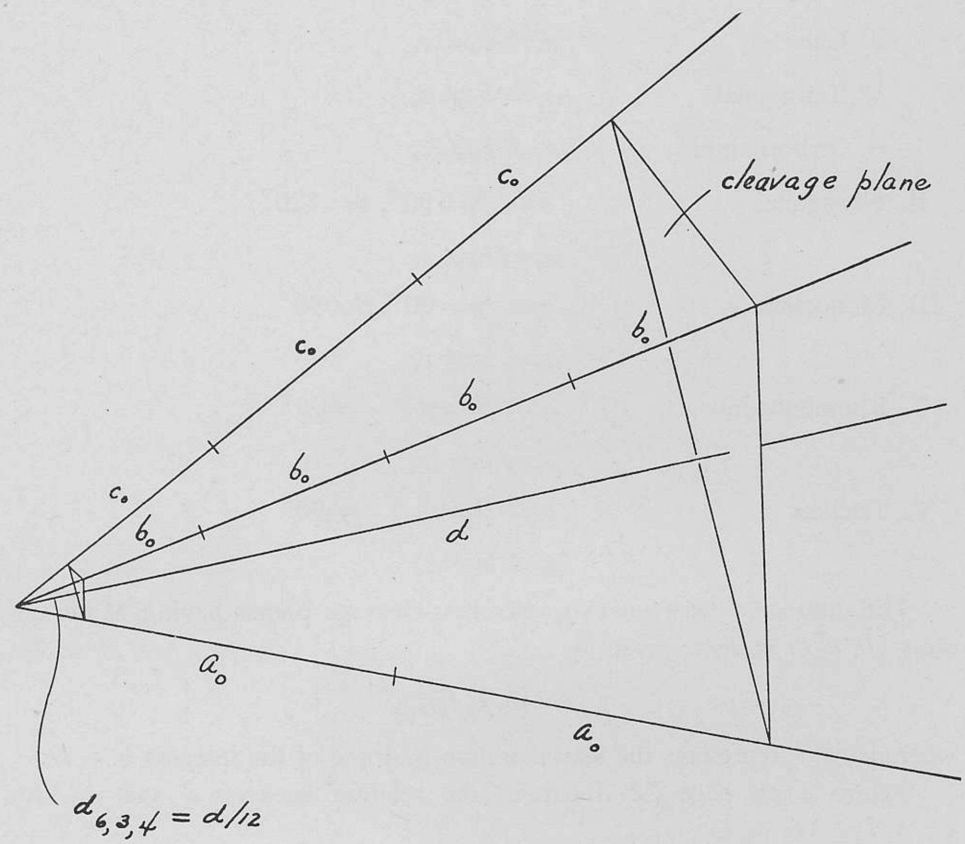

Figure 2

INSTITUTE OF NORTHWEST RESOURCES

The second Annual Institute of Northwest Resources will be held on the Oregon State College campus June 19–30, 1950, for teachers, graduate students, and other adults concerned with resource analysis, conservation, and development. The two-weeks institute will feature authorities on Northwest regional resources development and utilization in daily seminar discussions on topics of importance to the Northwest. Field trips will be made for observation of resource utilization. A number of scholarships will be available for teachers and other leaders who are deserving of assistance. For additional information persons interested should contact: J. Granville Jensen, Professor of Geography, Oregon State College, Corvallis, Oregon.