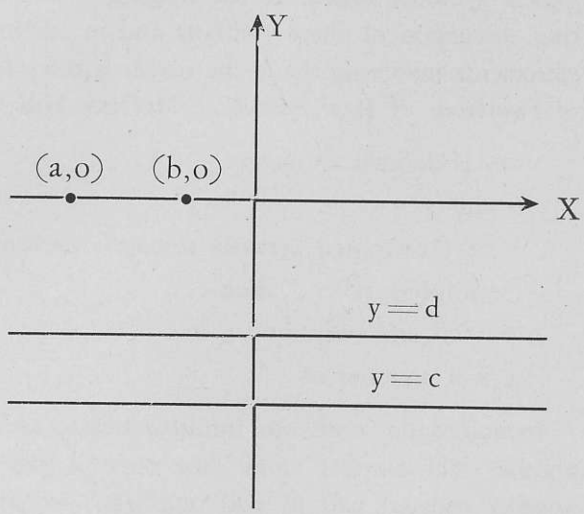
An Ordering of the Complex Numbers

A. E. LABARRE, JR.
Department of Mathematics
*University of Idaho**

QUITE FREQUENTLY the categorical statement is made that the set of all complex numbers is not an ordered set. This is not true, for the complex numbers do admit of ordering in quite the same sense that the set of all real numbers admit of ordering. It is true, however, that the system of complex numbers is not an ordered field, as we shall see. It is the purpose of this paper to define one of the possible orderings of the complex numbers and to discuss briefly some of its properties.

Some geometrical considerations will be helpful in understanding what follows. Let us assume, as in plane analytic geometry, that we have a rectangular coordinate system, that is, a one-to-one (1–1) correspondence between the points in a plane (XY) and the ordered pairs of real numbers (x,y). Under this bi-unique correspondence, each point which lies on the X-axis may be represented by an ordered pair of real numbers having the form $(x,0)$. We may regard this as a (1–1) correspondence between the set of all real numbers and the points on this line. If we think of the X-axis as being "generated" by a point P which



* Present address: Department of Mathematics, University of Oklahoma.

moves on this line in the positive direction, we may interpret the notion of "$b > a$" geometrically by saying that $b > a$ if and only if P "generates" a before it "generates" b or we might say $b > a$ if and only if b lies to the right of a. Inasmuch as the set of all real numbers R has this "natural" ordering, whose properties are well known, it would appear very desirable that any proposed ordering of the complex numbers C preserve the usual ordering of R. This is the primary motivation in the ordering presented here. In the sequel wherever convenient we shall assume that the (1–1) correspondence:

$$(x,y) \longleftrightarrow x + iy$$

between the points in the plane and the complex numbers has been established. In view of these considerations, it is natural in extending our notion of a point P as generating the real numbers as follows: we think of a point P as generating the ordered pairs of real numbers in the XY plane by moving from left to right along straight lines parallel to the X-axis. It only remains, then, to decide when given two distinct real numbers c and d whether our point P in traversing the one-parameter family of lines $y =$ constant generates the ordered pairs of real numbers belonging to $y = c$ or $y = d$ first. We arbitrarily agree that if $d > c$, then the line $y = c$ is generated before the line $y = d$. We now undertake to state these ideas in precise mathematical language. In order to accomplish this, we shall use certain logical symbols. Those logical symbols which we use frequently and their readings follow (for further discussion of these symbols and in particular the "punctuation" of our statements involving them, the reader is referred to L. M. Graves, *The Theory of Functions of Real Variables,* McGraw-Hill Book Co., Inc., 1946):

$\equiv$ is defined to mean

. and

‴ or (interposed between *mutually exclusive* statements)

$\rightarrow$ implies; if . . . , then . . .

$\longleftrightarrow$ if and only if

ϵ is a member of

In accordance with our intuitive notion of "one real number greater than another real number" and "one ordered pair of real numbers greater than another ordered pair of real numbers," we make the following definitions:

Definition 1.1. Let $x \epsilon R . x' \epsilon R$. *$x$ is greater than x'* (written $x > x'$) $\cdot \equiv \cdot$ there exists a positive number h such that $x' + h = x$.

Definition 1.2. Let $(x,y) \epsilon C$. $(x',y') \epsilon C$. *(x,y) is greater than (x',y')*

$$\text{(written } (x,y) > (x',y')) \cdot \equiv \cdot y > y' \vee y = y' . x > x' .$$

It is clear from these definitions that not only is the natural ordering of the real numbers preserved, but also the only numbers between a pair of distinct real numbers are real numbers. More precisely,

$$(x',0) > (x,y) > (x'',0) \cdot \rightarrow \cdot x' > x > x'' . \ y = 0 .$$

In the theorems which follow, certain familiar properties of the real numbers as well as the usual laws of multiplication, addition, etc., of complex numbers are assumed. It will be noted, as might very well be anticipated, that each theorem concerning a complex number depends upon the analogous theorem for a real number.

The Set of Complex Numbers as a Dense Linearly Ordered Set

The purpose of this section is to show that under the relation between complex numbers given in Definition 1.2, the set C is linearly ordered and dense. We begin by defining a linearly ordered set.

Definition 2.1. A *linearly ordered set* (written l.o.s.) is a system $(\Omega, >)$ (i.e., a set of arbitrary elements with a relation designated $>$) having the properties:

(01) $>$ is a relation on $\Omega\Omega$, i.e., it is defined for pairs of elements of Ω.

(02) $>$ is a transitive, i.e.,

$$\omega > \omega' . \omega' > \omega'' \cdot \rightarrow \cdot \omega > \omega'' .$$

(03) $\omega > \omega$ is true for no element of Ω, i.e.,

$$\omega \epsilon \Omega \cdot \rightarrow \cdot \omega \not> \omega .$$

(04) $\omega \neq \omega' \cdot \rightarrow \cdot \omega > \omega' \vee \omega' > \omega .$

Theorem 2.1. Under definition 1.2 together with the usual definition of equality of two complex numbers, namely:

$$(x,y) = (x',y') \cdot \equiv \cdot x = x' . y = y' ,$$

the set C is a l.o.s.

Proof:

(01) This is the deterministic property, i.e., for every pair of elements of C, it is determined whether $(x,y) > (x',y')$ or not. This follows as a consequence of the definitions and the fact that $>$ is a relation on RR.

(02) $(x,y) > (x',y') . (x',y') > (x'',y'') \cdot \rightarrow \cdot (x,y) > (x'',y'')$.

By hypothesis, we have

(A) $(x,y) > (x',y')$, i.e., $y > y' \vee y = y' . x > x'$, and
(B) $(x',y') > (x'',y'')$, i.e., $y' > y'' \vee y' = y'' . x' > x''$.

There are four cases which arise. They are:

(1) $y > y' . y' > y''$
(2) $y > y' . y' = y'' . x' > x$
(3) $y = y' . x > x' . y > y''$
(4) $y = y' . x > x' . y' = y'' . x' > x''$

In each case the conclusion follows easily from the definitions and the fact that the order relation for the reals has the property of the theorem.

(03) $(x,y) \in C \cdot \rightarrow \cdot (x,y) \ngtr (x,y)$.

Suppose the statement false, i.e., suppose there exists (x,y) such that $(x,y) > (x,y)$. It follows that $y > y \vee y = y \cdot x > x$, either of which is untenable since the order relation for the real numbers has the property (03).

(04) $(x,y) \neq (x',y') \cdot \rightarrow \cdot (x,y) > (x',y') \vee (x',y') > (x,y)$.

Since $(x,y) \neq (x',y')$, one of the following persists:

(1) $x = x' . y \neq y'$ (2) $y = y' . x \neq x'$ (3) $x \neq x' . y \neq y'$

In cases (1) and (2), the conclusion follows readily on invoking the property (04) for the reals; that same property yields the desired result in the last case when it has been subdivided into the four cases to which it gives rise.

Definition 2.2. A system $(\Omega, >)$ is said to be *dense* in case

$\omega \in \Omega . \omega' \in \Omega . \omega > \omega' : \rightarrow :$ there exists $\omega'' \in \Omega$ such that $\omega > \omega'' > \omega'$.

Theorem 2.2. The system $(C, >)$ is dense.

Proof: Let $(x,y) \in C$ and $(x',y') \in C$ be given such that $(x,y) > (x',y')$. Now $(x,y) > (x',y') \cdot \rightarrow \cdot y \geqq y'$, by definition 1.2. If $y > y'$, there exists y'' such that $y > y'' > y'$ since the system $(R, >)$ is dense. Hence the conclusion. In case $y = y'$, we must have $x > x'$, since $(x,y) > (x',y')$. But now the conclusion follows in the same way as before.

Further Properties of the System $(C, +, \times, >)$

It is a familiar theorem that the system $(C, +, \times)$ is a *field.* The system $(C, +, \times, >)$, however, is not an *ordered field* as will be seen presently. We begin by giving a list of categorical postulates for the system of real numbers, i.e., the system $(R, +, \times, >)$. This system has the following properties:

(R1) There exists $x \in R$. there exists $x' \in R$ such that $x \neq x'$, i.e., the class R contains at least two elements.

(R2) $(R, +)$ is a commutative group.

(R3) $\times$ is on RR to R, and is associative, commutative, and distributive with respect to $+$.

(R4) The class R with the identity 0 for addition omitted is a group under the operation $\times$.

(R5) $(R, >)$ forms a linearly ordered set.

(R6) $x > x' \,.\, x'' \cdot \rightarrow \cdot x + x'' > x' + x''$.

(R7) $x > 0 \,.\, x' > 0 \cdot \rightarrow \cdot x \times x' > 0$.

(R8) $(R, >)$ is complete, i.e., every subset K of R which has a lower bound has a greatest lower bound in R.

Definition 3.1. A system having the properties *(R1)* through *(R4)* is a *field;* if, in addition, it has the properties *(R5)* through *(R7)*, it is an *ordered field.* If it is an ordered field and satisfies *(R8)*, it is a *complete ordered field.*

It can be shown (see reference to L. M. Graves) that the only complete ordered field is the real number system, in the sense that any two systems having the properties *(R1)* through *(R8)* are simply isomorphic, i.e., a (1–1) correspondence betwen their elements can be defined so that it preserves the operations $+$ and $\times$ and the relation $>$. This is what we mean when we refer to the postulates above as being categorical. We have already stated that the system $(C, +, \times, >)$ is not an ordered field. This is clear since under definition 1.2, the requirement *(R7)* is not met—all we need do is observe that $(0,1) > (0,0)$ but $(0,1) \times (0,1) = (-1,0) \not> (0,0)$. This is by no means peculiar to the ordering which we have defined for we now show that there is no definition of "positive complex number" for which the system $(C, +, \times, >)$ is an ordered field. We now proceed to sharpen this statement. First, we recall that $-(x,y) = (-x, -y)$ since $(x,y) + (-x, -y) = (0,0)$. We now prove a lemma:

Lemma 3.1. Let there exist a relation "$>$" defined on CC having the property that

$$(i)\ (x,y) > (x',y') \,.\, (x'',y'') \cdot \rightarrow \cdot (x,y) + (x'',y'') > (x',y') + (x'',y''),$$

then $(x,y) > (0,0) \cdot \leftarrow\rightarrow \cdot (0,0) > -(x,y)$.

Proof: We are given $(x,y) > (0,0)$. Then by property (i), $(x,y) + (-x, -y) = (0,0) > (-x, -y)$. Conversely, given that $(0,0) > -(x,y)$, then by (i) we have $(0,0) + (x,y) > (-x, -y) + (x,y) = (0,0)$.

We are now in a position to prove the following theorem:

Theorem 3.1. There is no relation "$>$" on CC for which the system $(C, +, \times, >)$ is an ordered field.

Proof: Consider $(0,1) \in C$. Now $(0,1) \neq (0,0)$. Hence, by *(R5)*, we must have $(0,1) > (0,0)$ $^{\omega}$ $(0,0) > (0,1)$.
Suppose $(0,1) > (0,0)$. Then by the definition of multiplication in C and by *(R6)*,

$$(0,1)\ (0,1) = (-1,0) > (0,0)$$

in which we have omitted the sign for multiplication. By the same reasoning,

$$(-1,0)\ (0,1) = (0,-1) > (0,0)\ .$$

But $(0,1) > (0,0) \cdot \rightarrow \cdot (0,0) > (0,-1)$, by lemma 3.1. It follows that $(0,1) > (0,0)$ is untenable. We now assume the only alternative: $(0,0) > (0,1) = -(0,-1)$. According to lemma 3.1, $(0,-1) > (0,0)$.
Then by *(R6)*, we have $(0,-1)\ (0,-1) = (-1,0) > (0,0)$ and finally $(-1,0)\ (0,-1) = (0,1) > (0,0)$. A contradiction. This completes the proof of the theorem.

It is not difficult to show that *(R6)* is shared by the system $(C, +, \times, >)$. The proof is omitted. We now inquire as to the completeness of $(C, >)$ in the sense of *(R8)*. The following example shows that $(C, >)$ is not complete under our definition of $>$. Consider the set A of all points (x,y) *interior* to the circle defined by $x^2 + y^2 = 1$. Any point having the form (x,a) with $a \geqq 1$ will serve as an upper bound for A. However, there is no least upper bound.

In passing, it should be observed that the principle used in ordering the complex numbers can be easily extended to ordering n-tuples of real numbers. Furthermore, many extensions in complex variable theory are immediately suggested which have not been discussed here.