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Eigenvalue Analysis of an Elastostatic Sector Problem

Abtstract

Equations are derived for the stresses in a sector subjected to a prescribed loading on its circumferential edge. The radial edges are tractionless. An Airy stress function which satisfies both the governing differential equation and the boundary conditions is derived in terms of an infinite series of biharmonic (Fadle) eigenfunctions. Numerical results are presented for the 60° sector, but the method of solution is not restricted to the 60° sector and the specific circumferential loading discussed herein; the solution is applicable to an arbitrary loading on the circumferential edge of a sector of any angle. Eigenvalues for the 30°, 45°, 60°, 75°, 90°, and 120° sectors are presented, and the eigenvalue extraction method is discussed in detail in Appendix II.

Introduction

Other investigators have discussed eigenvalue extraction methods to obtain the roots of the characteristic equation (Eq. 7) discussed herein. In an early paper, Brahtz (1) discussed an eigenvalue solution in conjunction with his investigation of the stresses at the base of a gravity dam. The approximate eigenvalues he generated are given by Eqs. 10 of this paper. Horvay and Hanson (2) used a variational method to obtain approximate eigenvalues. More recently, Little and Thompson (3) used a Newton-Raphson iteration technique in the complex plane to determine the roots of Eqs. 7 of this paper. They used asymptotic expressions for the roots which were then used to obtain an initial guess for each eigenvalue.

In the analysis of the specific problem discussed herein, the author attempted to utilize the approximate eigenvalues determined by the above-mentioned authors but was unable to satisfy the boundary conditions on the radial edges of the sector when those approximate values were used. After the approximate roots (Eqs. 10) given by Brahtz were refined by the iterative method discussed in Appendix II and then used in the analysis, the author was able to satisfy the boundary conditions on the radial edges of the sector to six decimal places. The boundary conditions on these edges are quite sensitive to the accuracy of the eigenvalues. Approximate eigenvalues result in a large error in the boundary conditions on the radial edges; precisely determined roots are crucial to the analysis of the problem.

There are two requisites in the stress function method of analysis of a plane elastostatic problem: the Airy function, φ, must satisfy the governing equation, $\nabla^4\varphi = 0$, and the boundary conditions of the problem must be satisfied. The Airy function derived in this exercise meets both requirements.

Formulation of the Problem

The sector of Figure 1 has tractionless radial edges and the following loading on the circumferential boundary:

$$\left.\begin{aligned}\bar{\sigma}_r &= \frac{1}{R_o^2}\left[1-\frac{12\vartheta^2}{\gamma^2}\right]\\ \bar{\tau}_{r\vartheta} &= \frac{1}{R_o^2}\left[\vartheta-\frac{4\vartheta^3}{\gamma^2}\right]\end{aligned}\right\} \quad ——————— 1.)$$

These boundary loads can be shown to represent a system in equilibrium since they can be obtained from the Airy stress function

$$\varphi_1 = -\frac{\gamma^2}{16}\left[1-\frac{4\vartheta^2}{\gamma^2}\right]^2 \quad ——————————— 2.)$$

Since φ_1 does not satisfy the governing equation, $\nabla^4\varphi_1 = 0$, it cannot be used to determine the stresses at an interior point of the sector. The problem, therefore, is to determine a biharmonic function which furnishes the self-equilibrating loads (Eq. 1) on the circumferential edge and zero shear and tangential stresses on the radial boundaries.

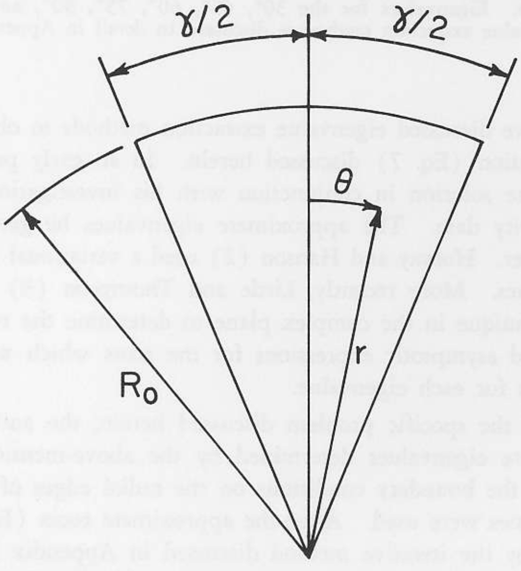

FIGURE 1. COORDINATE SYSTEM

Solution

Assume an Airy function, φ_2, represented as

$$\varphi_2 = r^{n+1}\, g(\vartheta) \quad ———— 3.)$$

in which n is a constant. Substitution of φ_2 into the governing differential equation, $\nabla^4\varphi_2 = 0$, yields

$$\frac{d^4g(\vartheta)}{d\vartheta^4} + 2(n^2+1)\frac{d^2g(\vartheta)}{d\vartheta^2} + (n^2-1)^2\, g(\vartheta) = 0 \quad ———— 4.)$$

which has the solution

$$g(\vartheta) = A_1\cos(n+1)\vartheta + B_1\sin(n+1)\vartheta + C_1\cos(n-1)\vartheta + D_1\sin(n-1)\vartheta \quad —— 5.)$$

in which A_1, B_1, C_1, and D_1 are constants.

On the tractionless radial edges of the plate, the boundary conditions are

$$\left.\begin{aligned} \left[\sigma_{\vartheta 2}\right]_{\vartheta = \pm\gamma/2} &= 0 \\ \left[\tau_{r\vartheta 2}\right]_{\vartheta = \pm\gamma/2} &= 0 \end{aligned}\right\} \qquad \text{6a.)}$$

Since $\sigma_{\vartheta 2} = \dfrac{\partial^2 \vartheta_2}{\partial r^2}$ and $\tau_{r\vartheta 2} = -\dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial \varphi_2}{\partial \vartheta}\right]$, the preceding boundary conditions become

$$\left.\begin{aligned} \left|g(\vartheta)\right|_{\vartheta = \pm\gamma/2} &= 0 \\ \left|\frac{dg(\vartheta)}{d(\vartheta)}\right|_{\vartheta = \pm\gamma/2} &= 0 \end{aligned}\right\} \qquad \text{6b.)}$$

Two equations result when $g(\vartheta)$ from Eq. 5 is substituted into Eqs. 6b:

$$A_1 \cos(n+1)\frac{\gamma}{2} + C_1 \cos(n-1)\frac{\gamma}{2} = 0$$

$$A_1(n+1) \sin(n+1)\frac{\gamma}{2} + C_1(n-1) \sin(n-1)\frac{\gamma}{2} = 0$$

To ensure that A_1 and C_1 both do not equal zero, the determinant of the coefficients of A_1 and C_1 in the preceding equations must vanish; therefore,

$$\begin{vmatrix} \cos(n+1)\dfrac{\gamma}{2} & \cos(n-1)\dfrac{\gamma}{2} \\ (n+1) \sin(n+1)\dfrac{\gamma}{2} & (n-1) \sin(n-1)\dfrac{\gamma}{2} \end{vmatrix} = 0$$

The preceding determinant reduces to the characteristic equation

$$n \sin\gamma = -\sin n\gamma \qquad \text{7.)}$$

In the interval $0° \leq \gamma \leq 180°$, the only real roots of Eq. 7 occur when $\gamma = 180°$ and are equal to $n = 1.00, 2.00, 3.00, \cdots$. The roots for all other γ in this range are complex and can be represented as

$$n = a_n + ib_n \qquad \text{8.)}$$

Substitution of Eq. 8 into Eq. 7 yields two equations, one from the imaginary terms and the other from the real terms. The equations are

$$\left.\begin{aligned} \sin a_n\gamma \cosh b_n\gamma &= \pm a_n \sin\gamma \\ \cos a_n\gamma \sinh b_n\gamma &= \pm b_n \sin\gamma \end{aligned}\right\} \qquad \text{9.)}$$

Eigenvalues (a_n and b_n) for the symmetric problem are associated with the negative sign on the right side of Eq. 9; the positive sign is for the case of an antisymmetric problem.

Brahtz developed the following equations for determining approximate eigenvalues:

$$\left.\begin{aligned} a_k &= \frac{\pi(2k+1) - \varepsilon_k}{\gamma^2} \\ b_k &= \frac{1}{\gamma} \log_e \left\{ \left[\frac{\pi(2k+1) - \varepsilon_k}{\gamma} \right] \sin\gamma \right\} \end{aligned}\right\} \quad ------ \quad 10.)$$

in which

$$\varepsilon_k = \frac{4 \log_e \left[\dfrac{\pi(2k+1) \sin\gamma}{\gamma} \right]}{\pi(2k+1)}$$

and

$$k = 1, 2, 3, \cdots$$

Manipulation of Eq. 9 yields

$$b_n \sin\gamma = \frac{\cos a_n\gamma}{\sin a_n\gamma} \left[a_n^2 \sin^2\gamma - \sin^2 a_n\gamma \right]^{1/2} \quad ------ \quad 11.)$$

By the iterative technique employed in the solution of Eqs. 11, the approximate values, a_k, rounded off to the nearest tenth's decimal place, are substituted into Eq. 11 for a_n. The tenth's decimal place of a_n is varied from 0 to 9, and each value of a_n generates a corresponding value of b_n by Eq. 11. That pair of a_n and b_n which best satisfies Eq. 11 is selected; a_n is substituted back into Eq. 11, and the procedure is repeated by varying the hundredth's decimal place of the previously refined value of a_n. Each successive iteration refines a_n and b_n to one more decimal place. The iterative procedure, in its entirety, is presented in Appendix II.

Since each sectorial angle, γ, has an infinite number of n's satisfying Eq. 7, the Airy function, φ_2, of Eq. 3 can be written as

$$\varphi = \sum_{n=1}^{\infty} K_n r^{n+1} \left[\cos(n-1)\frac{\gamma}{2} \cos(n+1)\vartheta - \cos(n+1)\frac{\gamma}{2} \cos(n-1)\vartheta \right] \quad -- \quad 12.)$$

in which

$$K_n = \frac{-C_1}{\cos(n+1)\dfrac{\gamma}{2}}$$

By the theory of elasticity, the known expressions for stress in terms of an Airy function are

$$\sigma_{r_2} = \frac{1}{r}\left[\frac{\partial \varphi_2}{\partial r}\right] + \frac{1}{r^2}\left[\frac{\partial^2 \varphi_2}{\partial \vartheta^2}\right]$$

$$\sigma_{\vartheta_2} = \frac{\partial^2 \varphi_2}{\partial r^2}$$

$$\tau_{r\vartheta_2} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial \vartheta_2}{\partial \vartheta}\right]$$

Since Eq. 12 is complex, differentiation of φ_2 in accordance with the preceding equations yields both real and imaginary components for the stresses. Only the real components are used, and they are

$$\left.\begin{aligned} R_0^2\sigma_{r_2} &= \sum_{n=1}^{\infty} \rho^{a_n-1}\left[K_{1n}(M'\Psi_{1n} - N'\Psi_{2n}) - K_{2n}(N'\Psi_{1n} + M'\Psi_{2n})\right] \\ R_0^2\tau_{r\vartheta_2} &= -\sum_{n=1}^{\infty} \rho^{a_n-1}\left[K_{1n}(M'\Gamma_{1n} - N'\Gamma_{2n}) - K_{2n}(N'\Gamma_{1n} + M'\Gamma_{2n})\right] \\ R_0^2\sigma_{\vartheta_2} &= \sum_{n=1}^{\infty} \rho^{a_n-1}\left[K_{1n}(M'X_{1n} - N'X_{2n}) - K_{2n}(N'X_{1n} + M'X_{2n})\right] \end{aligned}\right\} \text{— — — — — 13.)}$$

Eqs. 13 determine the stresses at any point in the sector. The unknowns in the preceding equations are K_{1n} and K_{2n}; the remaining quantities are all functions of a_n and b_n and are known as biharmonic eigenfunctions. In Eqs. 13, the following expressions are used: $\rho = r/R_0$; $M' = \cos(b\ln\rho)$; $N' = \sin(b\ln\rho)$; and

$$\Psi_{1n} = TA_n - UB_n + PE_n - QF_n \qquad \Gamma_{2n} = -VH_n - WG_n + RL_n + SJ_n$$
$$\Psi_{2n} = -UA_n - TB_n + QE_n + PF_n \qquad X_{1n} = -TA_n + UB_n - PC_n + QD_n$$
$$\Gamma_{1n} = -VG_n + WH_n + RJ_n - SL_n \qquad X_{2n} = TB_n + UA_n - PD_n - QC_n$$

in which

$$A_n = F_n{}^*Q_1 - E_n{}^*P_1 \qquad F_n = H_n{}^*T_1 - G_n{}^*U_1$$
$$B_n = E_n{}^*Q_1 + F_n{}^*P_1 \qquad R = \sin(a_n-1)\vartheta \cosh b_n\vartheta$$
$$C_n = E_n{}^*T_1 + F_n{}^*U_1 \qquad S = \cos(a_n-1)\vartheta \sinh b_n\vartheta$$
$$D_n = F_n{}^*T_1 - E_n{}^*U_1 \qquad V = \sin(a_n+1)\vartheta \cosh b_n\vartheta$$
$$E_n = G_n{}^*T_1 + H_n{}^*U_1 \qquad W = \cos(a_n+1)\vartheta \sinh b_n\vartheta$$

and

$$E_n{}^* = a_n^2 - b_n^2 + a_n \qquad H_n{}^* = 2a_nb_n - 3b_n$$
$$F_n{}^* = b_n + 2a_nb_n \qquad J_n{}^* = a_n^2 - b_n^2 - a_n$$
$$G_n{}^* = a_n^2 - b_n^2 - 3a_n \qquad L_n{}^* = 2a_nb_n - b_n$$

The unknowns K_{1n} and K_{2n} in Eqs. 13 can be determined from the following boundary conditions on the circumferential edge of the plate:

$$\left.\begin{aligned} \left[\sigma_{r_2}\right]_{r=R} &= \bar{\sigma}_r \\ \left[\tau_{r\vartheta_2}\right]_{r=R} &= \bar{\tau}_{r\vartheta} \end{aligned}\right\} \text{— — — — — — — — — 14a.)}$$

By Eqs. 1 and 13, the preceding boundary conditions become

$$\left.\begin{aligned} \sum_{n=1}^{\infty}(K_{1n}\Psi_{1n} - K_{2n}\Psi_{2n}) &= 1 - \frac{12\vartheta^2}{\gamma^2} \\ \sum_{n=1}^{\infty}(K_{1n}\Gamma_{1n} - K_{2n}\Gamma_{2n}) &= \vartheta - \frac{4\vartheta^3}{\gamma^2} \end{aligned}\right\} \text{— — — — — — 14b.)}$$

K_{1n} and K_{2n} were determined by the method of collocations. Five collocation points require five pairs of eigenvalues, and the expansion of Eqs. 14b with five pairs of eigenvalues yields ten equations in terms of ten unknown constants: K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, K_{21}, K_{22}, K_{23}, K_{24}, and K_{25}. A numerical solution was generated for the 60° sector utilizing the eigenvalues of Table 1.

TABLE 1. Eigenvalues for the equations
$\sin a_n\gamma \cosh b_n\gamma = -a_n \sin\gamma$,
$\cos a_n\gamma \sinh b_n\gamma = -b_n \sin\gamma$.

$\gamma = 30°$	a_1	a_3	a_5	a_7	a_9
	8.06296510	20.46721420	32.61272710	44.69125580	56.74125140
	b_1	b_3	b_5	b_7	b_9
	4.20286721	5.83660141	6.69310361	7.28117457	7.72996777
$\gamma = 45°$	a_1	a_3	a_5	a_7	a_9
	5.39052940	13.65142760	21.74607660	29.79731160	37.82998910
	b_1	b_3	b_5	b_7	b_9
	2.72040954	3.81446502	4.38632240	4.77858034	5.07802655
$\gamma = 60°$	a_1	a_3	a_5	a_7	a_9
	4.05932900	10.24572693	16.31416346	22.35138131	28.37518334
	b_1	b_3	b_5	b_7	b_9
	1.95204969	2.77796301	3.20778899	3.50239786	3.72707165
$\gamma = 75°$	a_1	a_3	a_5	a_7	a_9
	3.26501250	8.20425240	13.05627122	17.88475030	22.70303473
	b_1	b_3	b_5	b_7	b_9
	1.46666459	2.13337291	2.47825705	2.71431532	2.89423319
$\gamma = 90°$	a_1	a_3	a_5	a_7	a_9
	2.73959330	6.84513506	10.88555219	14.90789058	18.92231171
	b_1	b_3	b_5	b_7	b_9
	1.11902459	1.68163464	1.97019946	2.16733263	2.31746456
$\gamma = 120°$	a_1	a_3	a_5	a_7	a_9
	2.09413908	5.15173009	8.17576424	11.18950207	14.19854503
	b_1	b_3	b_5	b_7	b_9
	0.60458498	1.04933008	1.26896453	1.41792756	1.53105086

TABLE 2. Stresses along circumferential boundary of the 60° sector (starred values indicate collocation points).

ϑ	$\bar{\sigma}_r$	σ_{r_2}	$\bar{\tau}_{r\vartheta}$	$\tau_{r\vartheta_2}$
0°	1.000000	1.013840	0.000000	0.000000
*5°	0.916668	0.916659	0.084884	0.084838
*10°	0.666666	0.666674	0.155140	0.155150
*15°	0.250002	0.250001	0.196349	0.196334
*20°	−0.333334	−0.333344	0.193925	0.193937
*25°	−1.083330	−1.083317	0.133324	0.133324
30°	−2.000020	−2.055078	0.000000	0.000001

The Airy function, φ_2, was derived to reproduce the given tractions on the circumferential boundary. The values in Table 2 indicate that φ_2 reproduced the stresses along this edge of the sector within four decimal places of their true values. Table 3 indicates how well the stresses are reproduced between points of collocation. Using a truncated series with five pairs of eigenvalues (alternate pairs beginning with a_1 and b_1) and the equations for σ_{r_2} and $\tau_{r\vartheta_2}$ of Eqs. 13 in Eqs. 6a, the boundary conditions on the radial edges are satisfied to six decimal places.

Conclusions

A method for determining the Airy stress function for a sector with tractionless radial edges and a self-equilibrating loading on the circumferential boundary is presented as a series of biharmonic eigenfunctions. Since the Airy stress function satisfies both the governing elasticity equation ($\nabla^4\varphi = 0$) and the boundary conditions on the radial edges of the sector, the method of solution is applicable to a sector of any angle with an arbitrary, self-equilibrating load on the circumferential boundary. The eigenvalues in Table 1 apply to any symmetric sector problem involving tractionless radial edges. The iterative technique discussed in Appendix II can be used to determine the roots of Eqs. 9 for a sector of any angle, γ.

TABLE 3. Stresses at points of noncollocation along the circumferential boundary of the 60° sector.

ϑ	$\bar{\sigma}_r$	σ_{r_2}	$\bar{\tau}_{r\vartheta}$	$\tau_{r\vartheta_2}$
3°	0.9700	0.9750	0.0518	0.0514
8°	0.7866	0.7850	0.1297	0.1307
13°	0.4366	0.4370	0.1842	0.1834
18°	−0.0800	−0.0816	0.2010	0.2022
23°	−0.7633	−0.7637	0.1654	0.1659

Appendix I—Notation

a_n = eigenvalue representing the real part of n;
b_n = eigenvalue representing the imaginary part of n;
i = imaginary unit;
n = $a_n + ib_n$, root of characteristic equation;
R_o = radius of sector;
r = radial coordinate;
γ = sector angle;
ϑ = angular coordinate;
ρ = r/R_0, dimensionless radial coordinate;
σ_ϑ = tangential stress;
σ_ρ = radial stress;
$\tau_{r\vartheta}$ = shear stress;
φ = Airy stress function.

Appendix II—Eigenvalue Determination

The following FORTRAN program generates 12 pairs of eigenvalues, a_n and b_n, for $\gamma = 90°$. Eigenvalues for other sectors are determined by making the appropriate changes in the arguments of the trigonometric functions appearing in the program. Twelve approximate values of a_k from Eqs. 10, all rounded off to the nearest tenth's decimal place, are entered into the program under A(I). To obtain all of the eigenvalues, a second run of the program usually is necessary. In the second run, values of a_k rounded off to the nearest hundredth's decimal place are entered into the program, requiring the modification of AA(M,N). The modified expression becomes AA(M,N) =A(N)−(.1D0−.01D0*(XM−1.D0))/10.D0**I . The following notation is used: S=sinγ; SA=sina_nγ; CA=cosa_nγ; X=$a_n^2\sin^2\gamma - \sin^2 a_n\gamma$; Y=$\sqrt{X}$.

```
      DIMENSION A(12),B(12),R(12),AA(21,12),AB(21,12),BB(21,12),ER(21)
      READ (5,20)  (A(I), I=1,12)
   20 FORMAT (11F 7.2/F 7.2)
      PI = 3.14159265
      DO 100 N=1,12
      DO 100 I=1,6
      DO 50  M=1,21
      XM=M
      AA(M,N)= A(N)- (1.DO-.1DO*XM)/10.DO**I
      S= DSIN(PI/2.DO)
      CA= DCOS(AA(M,N)*PI/2.DO)
      SA= DSIN(AA(M,N)*PI/2.DO)
      X= AA(M,N)**2*S**2-SA*SA
      Y= DSQRT(X)
      BB(M,N)=CA*Y/(SA*S)
      HCB= DCOSH(BB(M,N)* PI/2.DO)
      AB(M,N)= DABS(SA*HCB/S)
   50 ER (M)= DABS(AA(M,N)-AB(M,N))
      DO 100  M=1,20
      R(N) = ER(M)-ER(M+1)
      IF  (R(N)) 100,100, 40
   40 R(N)=ER(M+1)
      A(N)=AA(M+1,N)
      B(N)=BB(M+1,N)
  100 CONTINUE
      WRITE (6,30) (A(I), B(I), R(I), I=1,12)
   30 FORMAT (3F 20.8)
      STOP
      END
```

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