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## Obtaining a Composite Volume-Age Function with the Gompertz Model<sup>2</sup>

### Abstract

A procedure for obtaining a single volume-age function for varying site and stand conditions and species is presented. This composite function involves the fitting of the Gompertz model and its coefficients as functions of stand density (average basal area measure), site index (height of dominant and codominants), and quantified species. This procedure is illustrated with data for five interior British Columbia species groups.

Comparison of the ages at which the maximum net mean annual increments calculated from the composite function and from individual models occur indicate they are not different.

### Introduction

Volume is one of the most important indices of productivity and is used in the formulation of forest management policies and in the assessment of silvicultural treatments. Volume-age relationships are known to follow the sigmoid shape, and their existence in the form of yield tables dates back to the eighteenth century (Spurr, 1952).

Quite commonly, several volume-age or other convenient relationships are expressed for different species under various site conditions. We present in this paper a procedure for developing a model common to all species for a range of site and stand density classes that is a composite-function.

Important attributes of any volume-age function which have to be considered include the maximum current annual increment (mcai), the maximum mean annual increment (mmai) and the ultimate limiting value. The mmai represents the point of intersection of the cai and mai curves, and is also the point of tangency of the sigmoid curve (yield curve) with the straight line through the origin. The age of the mmai coincides with the physical rotation, that is, the rotation yielding the largest volume of wood per unit area over the length of the period. This coincidence does not necessarily imply the maximum return on the investment. The mcai corresponds to the point of inflection on the yield curve.

The procedure to be described relies heavily on the individual's ability to describe certain conditions accurately. It would be advantageous if coefficients of the model had meaningful biological interpretations, but since the method is only that of mathematical

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<sup>2</sup>Developed with support of PC006, a grant to Dr. J. H. G. Smith of the University of British Columbia by the Productivity Committee of the British Columbia Forest Service. Data provided by the Inventory Division of the British Columbia Forest Service.

description, this requirement is not very essential. It is necessary to have the same model for all the different conditions.

The modified Gompertz function (Winsor, 1932; Grosenbaugh, 1965) is used to demonstrate the method in view of its ability to describe existing British Columbia Forest Service (B.C.F.S.) data accurately (Nokoe, 1974).

#### The Method

Four main stages (a - d) are recognized.

- a) The modified Gompertz function,

$$V = a e^{-c(x-g)}$$

is fitted to each species group data for every combination of site class and stand density. In this non-linear function,  $a$  is the ultimate limiting value,  $c$  is a coefficient specifying curve shape, and  $g$  is the point of inflection in years.  $V$  is net volume in cubic meters per ha (close utilization standard), and  $x$  is the age class average in years.

b) The coefficients estimated in the preceding stage are then expressed as functions of species, site index, and stand density, using either linear or non-linear least squares procedure. Site index is the average height of the dominant or codominant trees of a particular species group in a given locality at an index age. This stage involves the quantification of the species.

c) This stage is optional. It involves defining "common" site index and stand density values for all the species under consideration. Such constant values could lead to poorer yield estimates than if the actual values for each species group were used.

d) The composite function could then be reproduced as yield tables or graphs for each species group under the indicated common or actual site indices and stand densities.

#### Illustration

The objective of this illustration is to obtain a composite volume-age function by site index (SI), and stand density (SD) for five interior B.C. species groups (S), viz. Lodgepole pine (*Pinus contorta* Dougl.), Spruce (*Picea sitchensis* (Bong.) Carr), Aspen (*Populus tremuloides* Michx.) Douglas fir (*Pseudotsuga menziesii* (Mirb.) Franco), Balsam (*Abies balsamea* (L.) Mill.).

Fourteen sets of species group data with net volume per unit area ( $m^3/ha$ ), basal area per ha, age, and site indices (and classes), and grouped under five SD classes were available from the basic study by Smith (1973). The SD classes were 1-50, 51-80, 81-120, 121-150, 151+ percent of average basal area per hectare of trees 18 cm diameter at breast height (dbh) and larger. Each SD value may be obtained by any of the following procedures;

- (i) using the mid-points of the respective SD classes

$$(ii) SD_{ijs} = \frac{\sum_{k=1}^a BA_{ijk} s}{\sum_{k=1}^a BA_{.jks}} \times 100 \text{ percent}$$

$$SD_{ijs} = \frac{\sum_{k=1}^a BA_{ijk}}{\frac{1}{n} \sum_{k=1}^a \sum_{i=1}^n BA_{ijk}} \times 100 \text{ percent}$$

or (iii)  $SD_{ijs} = \frac{1}{a} \sum_{k=1}^a \frac{BA_{ijk}}{BA_{.jks}} \times 100 \text{ percent}$

for  $i=1$ , ..... ,  $n$  ( $n$  SD classes = 5 in our case)  
 $j=1$ , ..... ,  $m$  ( $m$  Site classes = 3 " " "  
 $k=1$ , ..... ,  $a$  ( $a$  Age classes = 12 " " "  
 $S=1$ , ..... , 1 (1 Species groups = 5 " " "

where  $BA$  is basal area, and  $SD_{ijs}$  is the average stand density for species  $S$ , Site class  $j$  and  $SD$  class  $i$  for all ages. The notation  $BA_{.jks}$  is the average basal area for species  $S$ , site class  $j$  and age class  $k$ ; that is,

$$\overline{BA}_{.jks} = \frac{1}{n} \sum_{i=1}^n BA_{ijk}$$

Procedure (i) is not appropriate for stand classes with unequal widths, and (ii) is based on a linear relationship of average basal area with age which is untrue. Procedure (iii) was used in this study because of the limitations of (i) and (ii).

Stage 1: Estimates of the Gompertz coefficients  $a$ ,  $c$ ,  $g$ , (available on request) were obtained for each set of data using the UBC-BMDX85 non-linear computer program. This program, which is an adaptation from the BMDX85 program of the UCLA Documentation Center, uses the modified Gauss-Newton iterative procedure described by Hartley (1961).

Stage 2: The five species were quantified as follows:

1—lodgepole pine, 2—spruce, 3—aspens, 4—Douglas fir, and 5—balsam. Thus the variable  $S$  took on values of 1 to 5, whole numbers only. Coefficient estimates  $a$ ,  $c$ ,  $g$  were then expressed as functions of  $S$ ,  $SD$ ,  $SI$  and their transformations using the linear regression program developed by A. Kozak, with the condition that each of the independent variables  $S$ ,  $SD$ , and  $SI$  occurred at least once in any of the three functions.

The results obtained from this example were of the following form:

eqn 1:  $a = f(S^3, S^4, SD^2, SD^3, SD^4, SI_1 * SD)$  ;  $R^2 = .9006$   $SE_E = 60.0m^3/ha$

eqn 2:  $c = F(S, S^2, S^3, S^4, SI, SD^2, SD^3, SD^4, SD * SI, SD^2 * SI)$  ;  $R^2 = .4996$   $SE_E = 0.0089$

eqn 3:  $g = f(S, S^2, S^4, SI, SI^2, SD)$  ;  $R^2 = .7585$   $SE_E = 8.6 \text{ years}$

These regressions were highly significant with fairly high coefficients of multiple determination;  $R^2 \cdot SE_E$  represents the standard error of the estimate of the dependent variable. The values of  $a$  from the various conditions used in obtaining the above functions

ranged from 43.3 to 746.6,  $c$  ranged from 0.013304 to 0.074360, and  $g$  was from 21.765 to 114.42. Thus the  $SE_E$  obtained from the functions of  $a$ ,  $c$ ,  $g$  could be regarded as fairly reasonable.

Stage 3: To compute final yields from the composite model, "common" values for site index and stand density were used. Instead of using these values, use could have been made of the separate species values as indicated in Tables 1 and 2. These tables were derived from the basic data of Smith (1973).

TABLE 1. Height of dominants and codominants in meters at age 100 (Site indices) for interior British Columbia species groups.

Code	Species group Name	Site indices		
		Site 1	Site 2	Site 3
1	Lodgepole pine	27.1	22.3	16.8
2	Spruce	29.6	23.5	16.8
3	Aspen	—	27.4	19.5
4	Douglas fir	30.5	22.9	19.5
5	Balsam	29.0	22.6	17.7
	Average	29.0	23.7	18.1
	"Common"	29	24	18

The large differences in the average SD values for SD class 151+ were due to the open nature of the class. Accordingly, two "common" values of 170 and 205 were allocated to the class.

Stage 4: Final yield tables were then computed for species 1—5, SD 30—205, for site indices 29.0, 24.0, and 18.0 using the modified Gompertz model,

Table 2. Stand density values in percentages.

Species	SD class SC	1-50			51-80			81-120			121-150			151+		
		1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	Lodgepole pine	38	33	30	68	67	66	100	102	96	132	134	136	180	193	225
2	Spruce	37	32	28	69	67	64	103	100	102	134	133	131	178	177	210
3	Aspen	—	33	29	—	67	69	—	102	95	—	133	144	—	175	203
4	Douglas fir	37	35	35	69	66	67	99	100	100	134	136	135	167	204	208
5	Balsam	40	34	33	64	69	66	97	99	98	131	135	139	185	175	203
	Average	38	33	31	68	67	66	100	101	98	133	134	137	178	185	210
	"Common"		30		65			100			135			170	205	

$$- (\text{eqn.2}) (x - \text{eqn.3})$$

- e

$$V = (\text{eqn.1}) e$$

Maximum mai values for each species were then computed using the common site index respective species site index value. Table 3 provides summaries of the results for lodgepole pine (SC 1, 2, 3) and aspen (SC 2, 3). The predicted volumes (V) varied for the two approaches (common versus species site index), but the ages (A) at which these maximum values were attained were similar.

### Conclusion

A method of obtaining a composite volume-age function has been presented. This method is simple and requires a minimum amount of time and computation. It is not

TABLE 3. Part summary of variable density estimates of yields\*, mean annual increment (mai), and ages at which maia are maximum by site and SD classes.

Species group	**	SD 30		65		100		135		170		205							
		A	V	A	V	A	V	A	V	A	V	A	V						
Lodgepole pine 1c		85	141	1.66	90	228	2.51	87	314	3.61	85	346	4.66	75	437	5.83	75	529	7.05
"	1	90	121	1.94	90	202	2.24	86	296	3.44	90	408	4.53	90	516	5.73	—	—	—
"	2c	110	146	1.33	100	194	1.94	100	276	2.76	95	348	3.66	83	368	4.43	80	404	5.05
"	2	110	93	0.84	106	151	1.42	103	221	2.15	96	286	2.98	108	407	3.77	—	—	—
"	3c	115	91	0.79	120	137	1.41	120	198	1.65	113	245	2.17	110	271	2.46	110	278	2.52
"	3	126	50	0.39	122	89	0.73	115	121	1.05	118	175	1.48	100	206	2.06	—	—	—
Aspen	2c	125	69	0.55	115	122	1.06	113	199	1.76	108	286	2.65	105	349	3.32	98	378	3.86
"	2	126	146	1.16	112	181	1.61	114	302	2.65	104	341	3.28	104	408	3.92	—	—	—
"	3c	116	22	0.19	120	59	0.49	126	115	0.91	120	169	1.41	117	202	1.73	118	209	1.77
"	3	109	28	0.26	126	101	0.80	128	136	1.06	92	151	1.64	114	242	2.12	—	—	—

\*\* c—composite site index value used, e.g. 1c etc; otherwise from specific data.

\* A is age in years; V is cubic meter per hectare to a close utilization standard, net for decay only; mai is the yield divided by the age at which annual growth culminates to the nearest tenth.

restricted to the Gompertz function. The only requirement is that the same model should be used in describing each species and/or site conditions. Such models should be able to fit the data adequately. The method may also be applied to a single species; in this case, "species" will no longer be a variable.

#### **Acknowledgments**

The authors thank the referees for their helpful comments.

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*Received March 9, 1977*

*Accepted for publication January 13, 1978*